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THE QUADRATIC GLASS CEILING AND ITS CONSEQUENCES

O TETO QUADRÁTICO DE VIDRO E SUAS CONSEQUÊNCIAS

EL TECHO DE CRISTAL CUADRADO Y SUS CONSECUENCIAS

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ABSTRACT

Over the past thirty years, the Algebra Project has worked with a diversity of stakeholders (parents, university partners, administrators, teachers, and their students) to raise the floor of math literacy for the most under-served students in US school systems. In the course of this work, we have come to recognize a glass ceiling that the most under-served students and their teachers must contend with. It has blocked access to topics that have historically been considered advanced, but which actually are accessible to students at the levels of Algebra I and Geometry. It has fuelled community debates around how much math is too much math for some students and not enough for others. In this article, we will examine a few examples from Algebra I and Geometry, which clearly show this accessibility. These examples demonstrate that an appropriate floor for math literacy in the 21st century needs to be reconstructed to account for the gap between what could be taught and what is taught in secondary mathematics.

Keywords: math literacy. accessibility. algebra. geometry. calculus.

RESUMO

Nos últimos trinta anos, o Projecto Álgebra trabalhou com uma diversidade de partes interessadas (pais, parceiros universitários, administradores, professores e seus alunos) para elevar o nível de alfabetização matemática para os alunos menos atendidos nos sistemas escolares dos EUA. No decorrer deste trabalho, chegamos a reconhecer um teto de vidro com o qual os alunos mais carentes e seus professores têm de lidar. Ele bloqueou o acesso a tópicos historicamente considerados avançados, mas que na verdade são acessíveis aos alunos dos níveis de Álgebra I e Geometria. Ele alimentou debates na comunidade sobre o quanto a matemática é matemática demais para alguns alunos e insuficiente para outros. Neste artigo, examinaremos alguns exemplos de Álgebra I e Geometria que mostram claramente essa acessibilidade. Esses exemplos demonstram que um piso apropriado para a alfabetização matemática no século 21° precisa ser reconstruído para dar conta da lacuna entre o que poderia ser ensinado e o que é ensinado na matemática secundária.

Palabras-chave: alfabetização matemática. acessibilidade. álgebra. geometria. cálculo.

RESUMEN

Durante los últimos treinta años, el Proyecto Álgebra ha trabajado con una diversidad de partes interesadas (padres, socios universitarios, administradores, maestros y sus estudiantes) para elevar el nivel de alfabetización matemática para los estudiantes más desatendidos en los sistemas escolares de EE. UU. En el curso de este trabajo, hemos llegado a reconocer un techo de cristal con el que deben lidiar los estudiantes más desatendidos y sus maestros. Ha bloqueado el acceso a temas que históricamente se han considerado avanzados, pero que en realidad son accesibles para los estudiantes en los niveles de Álgebra I y Geometría. Ha alimentado debates comunitarios sobre cuántas matemáticas son demasiadas para algunos estudiantes e insuficientes para otros. En este artículo, examinaremos algunos ejemplos de Álgebra I y Geometría, que muestran claramente esta accesibilidad. Estos ejemplos demuestran que es necesario reconstruir un piso apropiado para la alfabetización matemática en el siglo XXI para tener en cuenta la brecha entre lo que se podría enseñar y lo que se enseña en matemáticas secundarias.

Palabras clave: alfabetización matemática. accesibilidad. álgebra. geometría. cálculo.

Introduction

In a posthumously published editorial Bob Moses (2021), the founder of the Algebra Project, wrote:

> Amidst the planet-wide transformation we are undergoing, from industrial to information-age economies and culture, math performance has emerged as a critical measure of equal opportunity. We can see the collateral damage of inequities in math education in the way that students are tracked into dead-end math courses and how that tracking is then used to deny them other opportunities because they cannot demonstrate the required math competencies on standardized tests. Simply look at how the failure to complete math requirements is strongly correlated with not completing either high school or post-secondary education.

These inequalities in math education and their collateral damage affect the vast majority of American students, but their most severe impact is upon communities of color. While the American meritocratic perspective sees these inequalities as evidence that the equality of opportunity does not imply an equality of results, the literacy paradigm that the Algebra Project operates under views the inequality of results as implying an inequality of opportunities to learn. The purpose of this paper is to reveal one source of these inequalities affecting all students in the United States.

Within the Algebra Project, we have seen that these inequalities stand in plain sight but are not easily recognized because their origins lay deeply embedded in the presuppositions of our standards, policies, and practices. These presuppositions determine what we think is advanced mathematics versus elementary mathematics, what we think certain students can or cannot learn, and consequently, what we think certain students should or should not learn (Rosenstein, 2017; Thompson, 2008).

Presently the US math education research community has a major focus on examining how we teach and how students learn. Pedagogical content knowledge, PCK (Shulman, 1986), and mathematical knowledge for teaching, MKT (Ball, 1990) are two of the most prominent perspectives for how we teach. Constructivist research is focused on the mental construction of knowledge and understanding by students. This broad research agenda extends from cognitive constructivism (Clements, 1990) and neo-Piagetians such as APOS Theory (Dubinsky, 2001) to social constructivist (Van de Veer, 1994; Bransford, Brown & Cocking, 1990: Steffe, & D'Ambrosio, 1995). But these research frameworks take the underlying mathematics as given. They have not considered whether the mathematical content itself is problematic. Not only is *what we teach* as important as *how we teach,* but *how we teach* depends critically on *what we teach*. In this regard, the perspective we have developed in the Algebra Project suggests that significant portions of the learning issues faced by low-performing students in mathematics, particularly at the secondary level, are, in fact, didactic-genic, i.e., they have their origin, not in a lack of student preparedness or understanding, but rather in what we teach and how we teach it. The research agenda we are developing subjects the underlying mathematical content and the way in which that content is taught to a process of theoretical reconstruction and criticism. It is a preparatory study to PCK, MKT, and Constructivism in that it finds problematic and critically examines what these perspectives take as given. This article presents an example of the content analysis, theoretical reconstruction, and social criticism of the beginning high school mathematics sequence: Algebra I and Geometry. For students performing below proficiency in mathematics, and consequently, for all students, this analysis has the potential to reset the boundaries for what can be taught within a given content area and consequently what should be taught in that content area. To paraphrase Peter Gabriel Bergman in his introduction to *Basic Theories of Physics* (1949), the crucial task we are setting is to clarify the conceptual framework of the mathematics we teach and, in so doing to reforge it in order to keep abreast of societal need and educational progress.

The Quadratic Glass Ceiling we propose to uncover is the product of gaps, Blind Spots, in the mathematics that is taught for the first two years of high school in the US, namely in Algebra I and Geometry. The difficulty in recognizing these gaps is captured in the Einstein-Wertheimer Correspondence on Geometric Proofs and Mathematical Puzzles (Luchins, 1990). Albert Einstein suggests:

> Concepts that have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labeled as 'conceptual necessities,' 'apriori solutions,' etc. The road of scientific progress is frequently blocked for long periods by such errors. It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which this justification and usefulness depend.

Are these Blind Spots, these conceptual necessities, easily seen for what they are? No, not at all. For if they were, we wouldn't label them as Blind Spots. But we can perceive them if we first develop a picture of what should occupy these spots and then look again to see that these spots are empty. The Blind Spots creating the Quadratic Ceiling run through standards-based curricula, teacher mathematical and instructional practice, learning progressions and their associated tasks and products. In a discussion of learning progressions Maloney (2022) asserts

> … learning trajectories provide detailed descriptions of instructionally-grounded development of mathematical concepts and reasoning from the perspective of student learning, and, overall, building on decades of accumulated experience in mathematics education research. However, their greater importance may lie in their potential as frameworks that contribute an unprecedented coherence across classroom instruction, professional development, standards, and assessment, by focusing squarely on conceptual understanding and reasoning instead of assessment-driven procedural knowledge. This potential was sufficiently compelling as an organizing framework to have been cited as a basis for the Common Core mathematics standards, the new mathematics learning expectations that are consistent across most of the United States.

It, therefore, seems that the consequences of these Blind Spots and their associated Quadratic Glass Ceiling would follow a similar trajectory with parallel but detrimental impacts. Indeed, these gaps and ceilings produce an unprecedented incoherence across curricula, classroom instruction, professional development, learning progressions, and assessments, as the following examples will show.

An Example of a Blind Spot in High School Algebra I

The following two questions have been used in professional learning sessions with teams of high school mathematics teachers. The typical responses are given below.

- What is the meaning of the parameters m and b in the linear function,
  $y = mx + b$?
  m: the slope of the line,
  b: the y-intercept, $(0,b)$ the point where the line intersects the y-axis.

- What is the meaning of the parameters a, b, and c in the quadratic function, $y = ax^2 + bx + c$?
  a: $a > 0$, opens up, $a < 0$ opens down, $| a | > 0$ stretches, $0 < | a | < 1$ compresses.
  b: a horizontal translation of the parabola to the left or to the right, the line of symmetry is given by $x = - b/2a$
  c: the y-intercept

These responses are consistent with standard Algebra I texts. In the conventional formulation of Algebra I, the meaning of polynomial coefficients is never fully developed. The literature recognizes that this omission leads to certain misconceptions but has no notion of either a clear interpretation of all the coefficients or how to address the issue of their interpretation effectively. Ellis and Grinstead (2008) note that the equation for a parabola $y = ax^2+bx+c$, fosters two types of student misconceptions that they describe as "*b* is the slope of the graph" and "*a* is the slope of the graph." There is no recognition in their article that, indeed "*b*" is the slope of the graph of a sort (to be developed below). And regrettably, a conceptual misunderstanding is attributed to the student. This "misinterpretation on the part of the student" is a direct consequence of a gap in the mathematics of quadratic functions as usually developed. A standard task for students in Algebra I is to use a graphing calculator to determine the geometric meaning of the three parameters a, b, and c. The quadratic parameter "*a*" is recognized to determine whether the parabola opens up or opens down and how fast it opens. The constant term "*c*" is recognized as the y-intercept of the parabola. But the linear coefficient, b, seems to be problematic. The linear coefficient "*b*" does not have a clear interpretation, neither from teachers, math education researchers, in state curriculum standards, nor in the textbooks that are based upon those standards. Movshovtiz-Hadar (1993) describes an exercise where students graph each monomial of a quadratic: the squared term, the linear term and the constant term. But Movshovtiz-Hadar's exercise stops short of determining the relationship among the coefficients.

In our work with teachers and their students, we use the following exercise to resolve this indeterminacy. In addition to identifying the geometric meaning of each coefficient by observing the salient geometric feature affected by varying the coefficient's value, students are given the task of determining the effect of adding the next-order term starting with a constant polynomial and going up to a quadratic polynomial. The salient geometric feature of the constant monomial is its height above or below the x-axis. And that position is a global property of the graph. With the addition of a linear monomial, the global feature of the constant monomial becomes a local feature of the graph at the y-axis: the y-intercept of the graph. The salient feature of the linear polynomial is then the direction of the graph, again a global feature. With the addition of a quadratic monomial, the global feature of the linear monomial does not disappear any more than the y-intercept of the constant monomial disappears. The direction of the graph becomes a local feature of the graph at the y-intercept.

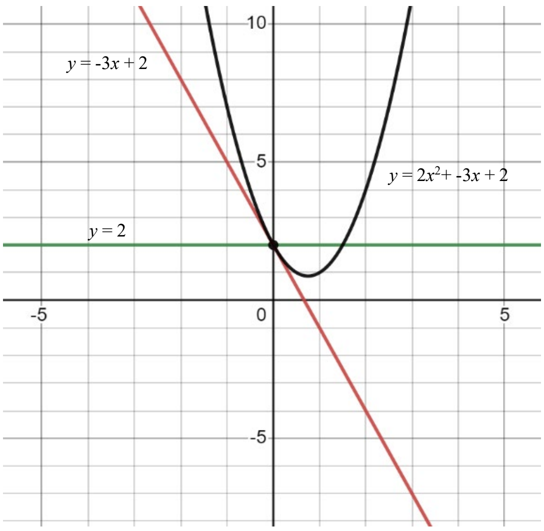


Figure 1

An example, shown in Figure 1, may help clarify this progression. We start with the constant monomial, $y = 2$, a horizontal line. The value of the constant function, 2, applies to all values of its domain. When the linear monomial is added to the constant term, the line with y-intercept 2 and slope -3 results. The slope characterizes the direction of the line and is a global property of the straight-line graph. Finally, a quadratic term is added to the linear polynomial. The quadratic coefficient, 2, indicated that the parabola opens upward and turns faster than the parent parabola, $y = x^2$. The linear polynomial now describes the direction of the graph at the y-intercept.

In general, students find that the highest-order coefficient describes a global property of the graph. Adding the next-order term causes the previous global feature to become a local feature at the *y*-intercept. Students determine the following results:

- *$y = c$: c describes the **height** of the graph.*
- *$y = bx + c$: c describes the **height** of the graph at the y-intercept, and b describes the **direction** of the graph and, consequently how fast the **height** of the graph is changing.*
- *$y = ax^2 + bx + c$: c describes the **height** of the graph at the y-intercept, b, the **direction** of the graph at the y-intercept, and a, the **turning** of the graph and consequently how fast the **direction** of the graph is changing.*

The linear portion of the quadratic polynomial at the *y*-intercept is traditionally called the tangent line. It allows students to "read off" the instantaneous rate of change of the polynomial function at the *y*-intercept. If this was known to teachers and their students, they could find the slope of the tangent line, the derivative, at any other position on the polynomial by simply translating the *y*-axis. That the linear portion of a polynomial corresponds to the tangent line at the *y*-intercept is not new. It is referenced, but not consistently, in practitioner journals. (Rabin, 2008;Carroll, 2009) Moreover, it is not systematically and coherently taught to student populations in our most under-served schools. Eureka, a Common Core-aligned curriculum, does mention the average rate of change across a parabola in its Algebra I curriculum (Eureka, 2015), but even there, the geometric meaning of the linear coefficient and its consequences is missing.

This example is the first instance of the Quadratic Glass Ceiling. The notion of the tangent line is an easily accessible concept within the framework of Algebra I. Its avoidance, as noted above, leads to several misconceptions on the part of students in their attempt to see some coherence and relationship among the mathematical concepts they are being taught.

An Example of a Blind Spot in High School Geometry

The following three questions have been used in professional learning sessions with teams of high school mathematics teachers. The typical responses are given below.

How is the area from $x = 0$ to $x = 1$ determined under

- a horizontal line given by the equation, $y = c$?
  Easy! The region under the line is a rectangle.
- a straight line given by the equation, $y = mx$?
  Easy! The region under the line is a triangle.
- a parabola given by the equation, $y = ax^2$?
  Wait. The parabola is curved. We need Calculus!

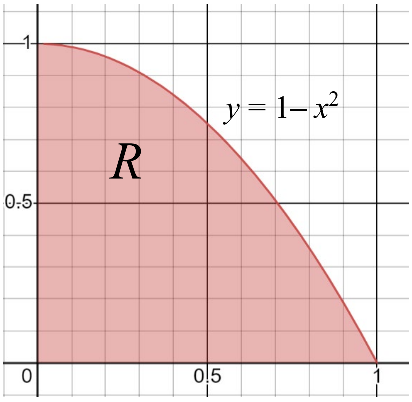
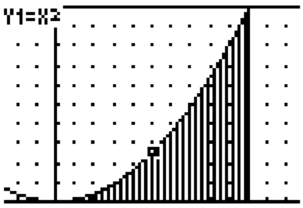
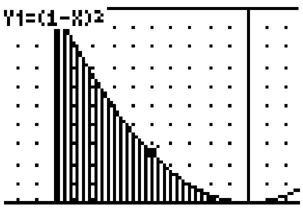


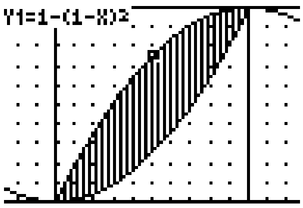
Figure 2

The conventional wisdom regarding the area under a parabola is typically framed along similar lines. We find in *University Calculus* (Hass, 2012):

> Suppose we want to find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$ and between the vertical lines $x = 0$ and $x = 1$. Unfortunately, there is no simple geometric formula for calculating the areas of general shapes having curved boundaries like the region R.

Despite this pronouncement of what cannot be done, the following exercise is a problem in high school Geometry requiring little more than the set of transformations that the Common Core Standards target as an essential component for learning Geometry. In this exercise, students are asked to complete a proof given a sequence of pictures. The third column is initially blank. In the third column, students are asked to explain how each picture relates to the following picture and how the shaded areas are connected. The problem is to determine the area of the parabolic triangle, P, in terms of the area of the enclosing square, S.

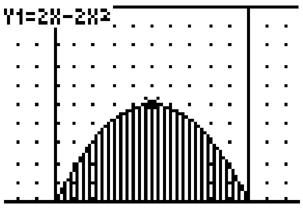
Geometry	**Algebra**	**Explanation of Picture**
 Picture 1	P	We want to find the area of the region between the parabola, $y = x^2$, and the positive x-axis within the unit square. We symbolize the area of this "triangular" region by P.
 Picture 2	P	The shaded figure is the same size and shape as the parabolic triangle in Picture 1. It is derived from the original parabolic triangle by a reflection about a vertical line at the base's midpoint.
 Picture 3	P	The shaded figure is the same size and shape as the original parabolic triangle. It is derived from the parabolic triangle in Picture 2 by a reflection about a horizontal line located at the midpoint of the vertical side of the square.



Picture 4

S* = 2*P* + *G

The area of the square (*S*) is equal to the area of the two parabolic triangles (*2P*) plus the area of the gap (*G*). The area of the gap is determined by the difference between its upper and lower boundaries.



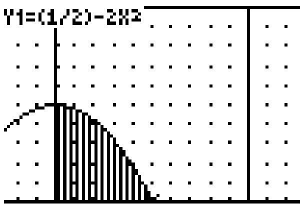
Picture 5

G

Since the area of the gap depends only upon the difference between the upper boundary (UB) and the lower boundary (LB),

A(UB – LB) = A(UB) – A(LB),

the area of the hill is equal to the area of the gap (*G*).



Picture 6

(1/2)*G*

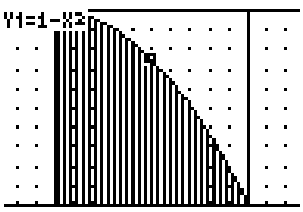
The hill is horizontally translated to the left by half the length of the square's base. The area of the shaded figure is equal to half the area of the gap.

Geometry	**Algebra**	**Explanation of Picture**

Picture 7

G

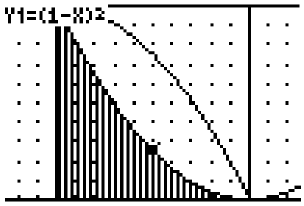
Horizontally scaling the figure in Picture 6 by a fact of 2 results in a figure with an area twice that of half the gap. The area of the shaded figure is simply *G*.



Picture 8

2*G*

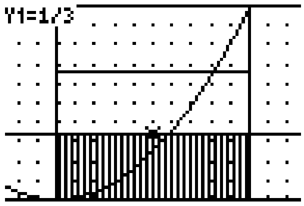
Vertically scaling the figure in Picture 7 by a factor of 2 results in a figure with twice the gap's area. The area of the shaded figure is 2*G*.



$G = P$

Picture 9

This figure is a reflection of the figure in picture 4, about a vertical line located at the midpoint of the base. Since the area of the previous figure was equal to twice the area of the gap (2*G*), the area of the lower triangle (*P*) must also equal the area of the gap (*G*).



$P = (1/3)S$

Picture 10

Therefore, the area of the square (2*P*+*G*) is equal to three times the area of the parabolic triangle, *P*. And the area of the parabolic triangle is one-third the area of the unit square.

So, referring back to Figure 2, there is *a simple geometric formula* for calculating the area of the region designated as *R*. We only need to read Pictures 8, 9, and 10 in reverse order to see that the area of *R* is equal to 2/3 of the enclosing square. The area under the graph of a cubic polynomial function entails a similar logic. The area under the n^{th}- order polynomial can be accomplished algebraically but requires the use of combinatoric identities beyond the scope of Algebra I or Algebra II.

The area under the parabola is typically the gateway problem to the Integral Calculus. As with the previous Algebra I exercise, we have successfully used this exercise with high school teachers and their Geometry students. Moreover, we see that not only is the conventional wisdom on the area problem mathematically incorrect but that it needlessly delays the introduction of concepts and techniques that are well within reach of students in a high school Geometry class.

Consequences of the Quadratic Glass Ceiling

The present situation with high school mathematics in the US is reminiscent of what Bob Moses referred to as sharecropper education. This was and still is an arrangement where communities are denied access to literacy levels within the educational system only to have those same levels of literacy used as conditions for blocking or enabling greater participation in the system. The data from the National Assessment of Education Progress (NAEP) Report for 2019, pre-COVID, provides a prime example of the effect on communities of color of this type of double bind. (NCES, 2019). The 12^{th}-grade NAEP mathematics achievement level results by race/ethnicity have 32% of white 12^{th} graders at or above proficient. NAEP defines proficiency as demonstrating "solid academic performance and competence in challenging subject matter." Blacks performed at 8%, Hispanics at 11%, and Native Americans at 9%. These results are a consequence of institutional constraints which perpetuate historical inequalities rather than simply the unpreparedness of the students from the communities involved. To expand upon the position stated in the Introduction, not only is *what we teach* as important as *how we teach* but *how we teach* depends critically on *what we teach and to whom*. What these students of color could have been taught and should have been taught they simply were not taught.

Calculus is not just a system of knowledge and techniques. It is also an institution. (Kaput, 2000). And as with many institutions, Calculus in the US educational system is a marker of privilege. The examples of the Quadratic Glass Ceilings given here demonstrate that there is no mathematical reason for the privileged access to Calculus that we presently see in US public schools. Instead, the reasons for restricting access are historical, social, and ultimately institutional. They serve to maintain an institutional system of privilege even after the original justifications no longer seem to be socially acceptable or politically relevant. This is the critical difference between institutional racism and its earlier explicit virulent variant. Under the aegis of meritocracy, the effects of institutional policies are claimed to be race-neutral. But the presence of institutional racism is found in its differential impact on communities of color as noted in the 2019 NAEP data.

The point of view developed here calls for a reassessment of the relationship between Algebra as the elementary discipline and Calculus as the advanced discipline. Alfred North Whitehead (1929) offers a clarifying perspective. "Algebra is the intellectual instrument for rendering clear the quantitative aspects of the world." Along these lines, Algebra can be understood in terms of its defining or foundational problems. Algebra addresses four fundamental problems: the problem of dependency, the problem of comparison, the problem of rate of change, and the problem of net change. The dependency problem asks, "How does one quantity depend upon another?" The answer to this question is typically a function, a concept often viewed as the central notion that Algebra should be built around. (Klein, 1932). The comparison problem asks, "When is one quantity or function less than, equal to, or greater than another?" The answer to this question typically involves the solution of equations and inequalities. The rate of change problem asks, "How fast is a quantity or function changing?" In Algebra I, this question is typically addressed by conditions of either constant rates of change or average rates of change. The net change problem asks, "By how much has a quantity or function changed?" And again, these problems are only considered for the cases of a constant rate of change or an average rate of change.

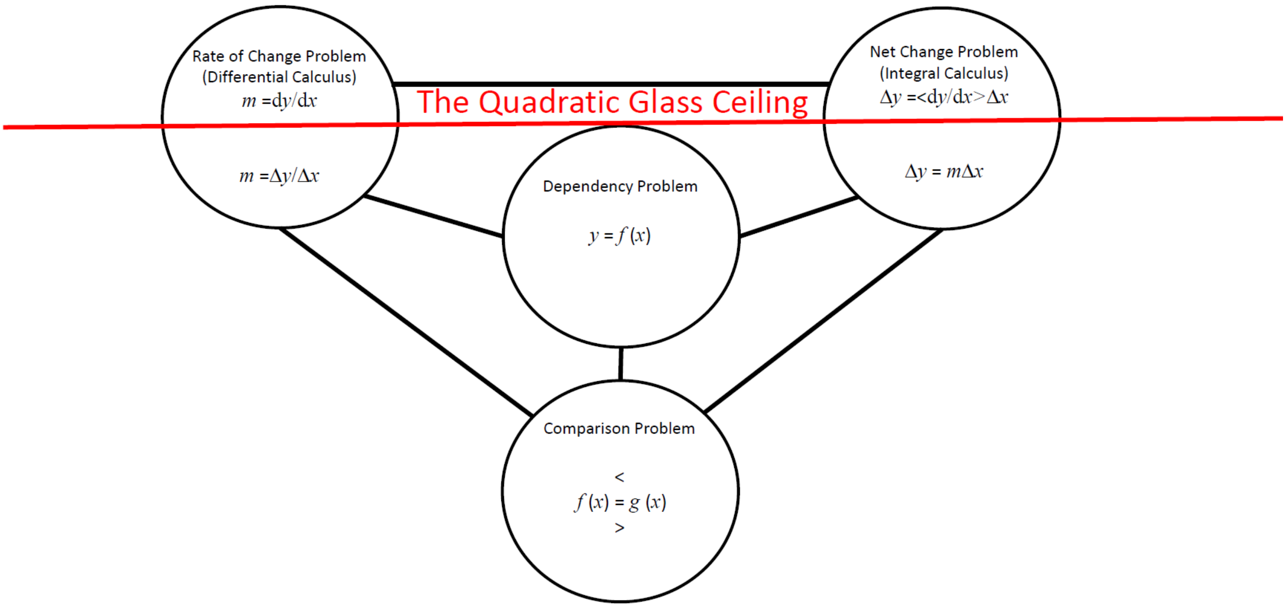


Figure 3

The preceding examples demonstrate that this restriction to constant and average rates of change is not a consequence of the mathematics itself. The notions of varying rates of change, slopes/derivatives, and their corresponding net changes, areas/integrals, are accessible through the study of polynomial functions, and in the first instance, through quadratic functions as early as Algebra I and Geometry. The present restrictions on their study are historically derived. And, as demonstrated above, the reason for their continued use in the restricted sense is an institutional choice, not a mathematical requirement.

At present in the US, Calculus has assumed a position as the capstone course of the high school mathematics sequence. Access to Calculus is presently based on a meritocratic rationale. According to the conventional wisdom, Calculus as an advanced topic requires four or more years of preparation for its acquisition: Algebra I, Geometry, Algebra II, Trigonometry, Pre-Calculus, and only then Calculus. (Almora Rios, 2023) The previous examples in this article demonstrate that Calculus is not the advanced topic it is presently assumed to be because Calculus is not synonymous with Analysis, the mathematics of approximation and limits. The main results of the Calculus can be derived from the advanced topic of limits or as shown above from an application of basic Algebra and Geometry. We have presented only the first steps in such a derivation here. Because the main concepts and techniques of Calculus fall squarely within the domains of Algebra I and Geometry, the present meritocratic prerequisites serve only to support the privilege of some and retard the advancement of others. In effect, the most practical prerequisite for Calculus with Limits is Calculus itself as a fundamental problem domain of introductory Algebra and Geometry. Calculus, in spite of the way it is presently taught, is nothing more than quantitative reasoning: reasoning in terms of quantity, rate of change, and net change. Or as Whitehead reminded us, "Algebra is the intellectual instrument for rendering clear the quantitative aspects of the world."

If we can indeed see these Blind Spots and the Quadratic Glass Ceiling they create, an appropriate floor for math literacy in the 21st century needs to be reconsidered and reconstructed to account for the gap between what could and should be taught and what is taught in secondary mathematics to all communities in the US.

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